SOLUTIONS OF EXERCISES.

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172

FIND the deflection of a homogeneous elastic beam of length $2v$, loaded uniformly, and supported at two points distant u from its middle point.

[*W. M. Thornton.*]

SOLUTION.

The bending moment for load w per inch is, at x inches from midspan,

$$M = \tfrac{1}{2} w (v^2 - 2uv + x^2);$$

whence for the differential equation to the elastica we have

$$EI \frac{d^2y}{dx^2} = \tfrac{1}{2} w (v^2 - 2uv + x^2).$$

The first integration gives

$$EI \frac{dy}{dx} = \tfrac{1}{2} w (v^2 x - 2uvx + \tfrac{1}{3} x^3),$$

no constant being added because the declivity of the elastica is zero at midspan. The second integration gives

$$EIy = \tfrac{1}{2} w \left(\frac{v^2 x^2}{2} - uvx^2 + \tfrac{1}{12} x^4 \right).$$

The deflection is found by making $x = u$;

$$\therefore\ d = \tfrac{1}{24} \frac{w}{EI} (u^4 - 12u^3 v + 6u^2 v^2).$$

[*W. M. Thornton.*]

208

PROVE that the surface of an oblate spheroid whose major semiaxis is a and eccentricity e, is equal to

$$2a^2\pi\left[1 + \text{Naperian log}\left(\frac{1+e}{1-e}\right)^{\frac{1-e^2}{2e}}\right].$$

[*R. S. Woodward.*]

SOLUTION.

The surface of the spheroid generated by $a^2y^2 + b^2x^2 = a^2b^2$ when it is rotated about its minor axis is 2π times

$$\int xds = \int\sqrt{x^2 + \frac{a^4}{b^2}y^2}.dy = \int\frac{a^2}{b^2}\sqrt{(1-e^2) + e^2y^2}.dy.$$

From this, by the ordinary integration-process, the result follows.

[*W. O. Whitescarver; Artemas Martin; J. E. Hendricks.*]

215

Two equal circles, radii r, intersect; find the average area common to both. [*Artemas Martin.*]

SOLUTION.

Let A and B be the centres of the two circles; C and E the intersections of their circumferences with one another; F and G their intersections with AB; D the intersection of AB and CE.

While the centre A is fixed, the centre B will be at the distance $2x$ from A if it is anywhere on the circumference described from centre A with radius $AB = 2x$. Hence, for every value of x, B can have $4\pi x$ positions.

Now $CD = \sqrt{(r^2 - x^2)}$, arc $CE = 2r\cos^{-1}\left(\frac{x}{r}\right)$; sector $BCFE = r^2\cos^{-1}\left(\frac{x}{r}\right)$, triangle $CBE = x\sqrt{(r^2 - x^2)}$, segment $CFED = r^2\cos^{-1}\left(\frac{x}{r}\right) - x\sqrt{(r^2 - x^2)}$, and the area common to both circles is

$$2r^2\cos^{-1}\left(\frac{x}{r}\right) - 2x\sqrt{(r^2 - x^2)} = S.$$

The *average* area required is

$$A = \int_0^r S \times 4\pi x dx \div \int_0^r 4\pi x dx,$$

$$= 4\int_0^r x\cos^{-1}\left(\frac{x}{r}\right)dx - \frac{4}{r^2}\int_0^r x^2\sqrt{(r^2 - x^2)}\,dx,$$

$$= \tfrac{1}{4}\pi r^2.$$

[*Artemas Martin.*]

216

Two equal spheres, radii r, intersect; find the average volume common to both. [*Artemas Martin.*]

SOLUTION.

Let $2x = AB =$ the distance between the centres of the spheres.

While the centre A is fixed, the centre B will be at the distance $2x$ from A if it is anywhere on the surface of the sphere whose centre is at A, and radius $AB = 2x$. Hence, for every value of x, B can have $16\pi x^2$ positions.

Now

$$CD = \sqrt{(r^2 - x^2)}, \quad FD = DG = r - x;$$

volume of segment

$$CGED = \tfrac{1}{2}(r - x) \times \pi(r^2 - x^2) + \tfrac{1}{6}\pi(r - x)^3,$$

and volume $CFEGC$, the volume common to both spheres,

$$= \pi(r - x)(r^2 - x^2) + \tfrac{1}{3}\pi(r - x)^3 = V.$$

The *average* volume required is

$$A = \int_0^r V \times 16\pi x^2 dx \div \int_0^r 16\pi x^2 dx,$$

$$= \frac{\pi}{r^3}\int_0^r [3(r - x)(r^2 - x^2) + (r - x)^3] x^2 dx,$$

$$= \frac{\pi}{r^3}\int_0^r (4r^3 - 6r^2 x + 2x^3) x^2 dx,$$

$$= \tfrac{1}{6}\pi r^3.$$

[*Artemas Martin.*]

218

THE length of none of three lines exceeds a; find the probability that an acute triangle can be formed with them. [*Artemas Martin.*]

SOLUTION.

Let x, y, and z denote the lengths of the lines, x being the greatest and z the least.

In order that the triangle may be acute, we must have $z > \sqrt{(x^2 - y^2)}$.

$$\therefore\ p = \int_0^a\int_0^x\int_{\sqrt{(x^2-y^2)}}^x dxdydz \div \int_0^a\int_0^x\int_0^x dxdydz,$$

$$= \frac{3}{a^3}\int_0^a\int_0^x\int_{\sqrt{(x^2-y^2)}}^x dxdydz,$$

$$= \frac{3}{a^3}\int_0^a\int_0^x [x - \sqrt{(x^2-y^2)}]\,dxdy,$$

$$= \frac{3}{a^3}\int_0^a (1 - \tfrac{1}{4}\pi)\,x^2dx = 1 - \tfrac{1}{4}\pi.$$

[*Artemas Martin.*]

229

IF $a^2 = b^2 - bc$ and $b^2 = c^2 - ac$, then $c^2 = a^2 - ab$. [*Frank Morley.*]

SOLUTION.

Eliminating a, b, and c, respectively, from the given equations, we obtain

$$(b-c)\,(b^3 + b^2c - 2bc^2 - c^3) = 0, \tag{1}$$

$$a\,(c^3 + c^2a - 2ca^2 - a^3) = 0, \tag{2}$$

$$a\,(a^3 + a^2b - 2ab^2 - b^3) = 0. \tag{3}$$

The symmetry of the right hand factors show that $c^2 = a^2 - ab$ is satisfied for three values of (1), (2), and (3), and also, for $a = b = c = 0$; but for $a = 0$, $b = c =$ a finite quantity, the given equations are satisfied, but $c^2 = a^2 - ab$ is not satisfied. The proposition is true for every other case, but fails in this particular one. [*E. Frisby.*]

230

SOLVE the equations

$$x^2(y - z) = a,$$

$$y^2(z - x) = b,$$

$$z^2(x - y) = c.$$

[*Frank Morley.*]

SOLUTION.

Let $y = vx$, $z = wx$; then the three given equations become

$$x^3(v - w) = a,$$

$$v^2x^3(w - 1) = b,$$

$$w^2x^3(v - 1) = -c.$$

$$\therefore \quad \frac{v^2(w-1)}{v-w} = \frac{b}{a},$$

$$\frac{w^2(v-1)}{v-w} = -\frac{c}{a}.$$

Subtracting the latter from the former, we have

$$vw - (v + w) = \frac{b+c}{a}.$$

Substituting this value of vw in either of the two equations just written, we have

$$(a + c)v + (a + b)w = -(b + c),$$

or
$$w = -\frac{(a+c)v + (b+c)}{a+b}.$$

$$\therefore \quad v^2 + \frac{2b}{a+c}v = -\frac{b(b+c)}{a(a+c)},$$

$$v = -\frac{b}{a+c} \pm \frac{1}{a+c}\sqrt{-\frac{bc}{a}(a+b+c)};$$

$$\therefore \quad w = -\frac{c}{a+b} \mp \frac{1}{a+b}\sqrt{-\frac{bc}{a}(a+b+c)}.$$

From the first of the three given equations, we obtain

$$x = \left(\frac{a}{v-w}\right)^{\frac{1}{3}};$$

$$\therefore \quad x = \left(\frac{a}{-\dfrac{b}{a+c} + \dfrac{c}{a+b} \pm \left(\dfrac{1}{a+c} + \dfrac{1}{a+b}\right)\sqrt{-\dfrac{bc}{a}(a+b+c)}}\right)^{\frac{1}{3}}.$$

[*R. A. Harris.*]

231

IF we take products of n consecutive terms of the arithmetical series a, $a - d$, etc., commencing for the first product with the first term, for the second product with the second term, and so on; and then multiply these products by the coefficients in the expansion of $(1 - x)^n$, the aggregate will be $n!\, d^n$, which is independent of the first term. [*W. W. Johnson.*]

SOLUTION.

1. For any series $a_0, a_1, a_2, \ldots$, the first term in the nth order of differences is equal to

$$a_n - na_{n-1} + \frac{n(n-1)}{1\,.\,2}\, a_{n-2} + \ldots + (-1)^n a_0.$$

2. If a_r be the rational function $Ar^n + Br^{n-1} + \ldots$, it is easy to see that the terms of the nth order of differences will be independent of r, and each equal to $n!\, A$.

3. The series formed as stated in question will have its $(r+1)$th term equal to

$$(a - rd)(a - \overline{r+1}\,.\,d) \ldots (a - \overline{r+n-1}\,.\,d),$$

which is of the form $(-d)^n . r^n + \ldots$; whence, by (1) and (2), we have

$$a_n - na_{n-1} + \ldots + (-1)^n a_0 = n!\,(-d)^n;$$

and, dividing by $(-1)^n$, we get the required result, viz.:

$$a_0 - na_1 + \frac{n(n-1)}{1\,.\,2}\, a_2 + \ldots + (-1)^n a_n = n!\, d^n,$$

wherein $a_0, a_1, a_2, \ldots a_n$ are any $n + 1$ consecutive terms of the series.

4. It may be added that if the coefficients of $(1 - x)^m$ be applied to any $m + 1$ consecutive terms of the series, the aggregate will be *zero* when $m > n$; and similarly for any series in which a_r is a rational nth-degree function of r.

[*James McMahon.*]

232 and 233

IF a square be inscribed in the face of a cube, the plane determined by one side and the corner of the opposite face corresponding to the adjacent corner of the same face touches the inscribed sphere. [*T. M. Blakslee.*]

SOLUTION.

Take the centre of the cube as origin of co-ordinates. The plane required passes through the points

$$(a, 0, a);\quad (0, a, a);\quad (a, a, -a),$$

and hence has for equation

$$2x + 2y + z = 3a,$$

and for distance from the origin $\qquad r = a.$

There are twenty-four such planes. [*T. U. Taylor; H. B. Newson.*]

236

If p, q, r, s are the lengths, supposed unequal, of the sides of a quadrilateral, prove that

$$[(p+q)(p+r)(p+s)(q+r)(q+s)(r+s)]^2$$
$$> [(p+q+r-s)(p+q-r+s)(p-q+r+s)(-p+q+r+s)]^2.$$

[*R. H. Graves.*]

The restrictions are altogether unnecessary. p, q, r, s may be any numbers whatever, positive, negative, or even 0, provided one at least is different from the others; and even when they are all equal, it is only a limiting case. If the sign $>$ means "not less than," the proposition is absolutely universal.

PROOF.

$$(p+q)^2 > (p+q)^2 - (r-s)^2, \text{ or } > (p+q+r-s)(p+q-r+s);$$
$$(r+s)^2 > (r+s)^2 - (p-q)^2, \text{ or } > (r+s+p-q)(r+s-p+q).$$

Forming the other squares in the same way, and multiplying them together, the proposition results. [*E. Frisby.*]

237

The axes of an ellipse are given, and one focal distance of a point on the curve. Find the ordinate of the point drawn to the major axis.

[*O. L. Mathiot.*]

SOLUTION.

$$r = \frac{b^2}{a + \sqrt{(a^2 - b^2)}.\cos v}, \quad y = r \sin v;$$

or

$$\sqrt{a^2 - b^2}.\, r \cos v = b^2 - ar,$$

$$\sqrt{a^2 - b^2}.\, r \sin v = \sqrt{a^2 - b^2}.y;$$

whence, by squaring and adding, we have

$$(a^2 - b^2)r^2 = (a^2 - b^2)y^2 + b^4 - 2ab^2r + a^2r^2,$$

$$y^2 = \frac{2ab^2r - b^2(b^2 + r^2)}{a^2 - b^2}.$$

[*E. Frisby.*]

238

LET P be a point on an equilateral hyperbola. Find the locus of the point of intersection of the ordinate of P and the perpendicular bisector of the longer of the supplemental chords drawn to P. [*R. H. Graves.*]

SOLUTION.

It can be easily proved that a rhomb may be constructed having for one diagonal the chord and the ends of the other diagonal on the axes. The equation to this diagonal (the perpendicular bisector) is evidently

$$\frac{x}{x' - a} + \frac{y}{y'} = 1, \text{ where } x'^2 - y'^2 = a^2.$$

Eliminate from these equations and $x = x'$ the co-ordinates x' and y'. The equation to the required locus is thus found to be

$$(x - a)^2y^2 - a^2x^2 + a^4 = 0.$$

[*R. H. Graves.*]

239

FIND the equation to QR of Exercise 205, P being given.

[*R. H. Graves.*]

SOLUTION.

Let the equation to the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and let (x', y') be the co-ordinates of the point P.

QR passes through $\left(\frac{a^4}{(a^2 - b^2)x'}, \frac{-b^4}{(a^2 - b^2)y'}\right)$. (See solution to Exercise 205.) QR also passes through the point $\left(-\frac{a^2}{x'}, 0\right)$. For, "If two chords of a

conic be drawn through two points on a diameter equidistant from the centre, any conic through the extremities of these chords will be cut by that diameter in points equidistant from the centre." (See Smith's Conic Sections, p. 202.)

Therefore the equation to the line QR is

$$\frac{xx'}{a^2} + \frac{yy'}{kb^2} = -1 \text{ (1), where } k = \frac{1-e^2}{1+e^2}.$$

COR. (1) touches $\frac{x^2}{a^2} + \frac{y^2}{k^2b^2} = 1$ at $(-x', -ky')$.

This is a special case of Ex. 3 of Art. 272, Salmon's Conic Sections.

REM. A similar investigation would apply to the hyperbola.

[*R. H. Graves.*]

241

AN ellipse cuts a confocal lemniscate where its ordinate is a maximum. Find the eccentricity of the ellipse. [*R. H. Graves.*]

SOLUTION.

Use the equations to the curve in vectorial co-ordinates,

$$r + r' = 2a \text{ (1) and } rr' = a^2e^2 \text{ (2)}.$$

At the points of intersection of (1) and (2)

$$r = a + b \text{ and } r' = a - b.$$

When the ordinate is a maximum it is evident that r and r' are at right angles. Then,

$$4a^2e^2 = (a+b)^2 + (a-b)^2;$$

$$\therefore \quad a^2 = 3b^2, \text{ or } e = \sqrt{\tfrac{2}{3}}.$$

[*R. H. Graves.*]

243

A NODAL quartic passes through the twelve intersections of three conics. Show that the polars of the node with regard to the conics meet in a point.

[*Frank Morley.*]

SOLUTION.

Any quartic passing through the twelve intersections of the conics $u = 0$, $v = 0$, $w = 0$, can be represented by the equation

$$fvw + guw + huv = 0,$$

for two other conditions would fix the quartic, and would be just sufficient to determine $f : g : h$,

If the quartic be nodal, the co-ordinates x, y, z of the node satisfy the three derivatives as to x, y, z; then, eliminating f, g, h, the equation to the locus of the node is

$$\begin{vmatrix} v_1w + vw_1 & u_1w + uw_1 & u_1v + uv_1 \\ v_2w + vw_2 & u_2w + uw_2 & u_2v + uv_2 \\ v_3w + vw_3 & u_3w + uw_3 & u_3v + uv_3 \end{vmatrix} = 0,$$

wherein the subscripts denote differentiation as to x, y, z, respectively. This equation easily reduces to

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0,$$

which represents the Jacobian of the three conics, and is evidently the locus of a point whose three polars $xu_1 + yu_2 + zu_3 = 0$, etc. meet in a common point. Hence the theorem. [*James McMahon.*]

244

If an oval of Cassini, confocal with a meridian of the earth, cuts it at a certain point, the angle at which the curves intersect is equal to the reduction of latitude at that point. [*R. H. Graves.*]

SOLUTION.

The focal radii at the point make equal angles with the central radius and the normal of the oval*; and they are equally inclined to the normal to the meridian. Hence the truth of the proposition. [*R. H. Graves.*]

245

The probability that an event A happens is p_1; the probability that an event B happens is p_2; the probability that neither happens is p_3; required, the probability that both happen. [*L. M. Hoskins.*]

SOLUTION.

Regarding the occurrence of the events A and B there are only four possible cases:—

1. A and B both happen;
2. A happens but not B;
3. B happens but not A;
4. Neither A nor B happens.

*See Williamson's Diff. Calc., Art. 193, Ex. 2.

Let a, b, c, d represent the probabilities of these several cases. The four cases being *mutually exclusive*, we must have

$$a + b + c + d = 1.$$

Also, each of the first two cases involves the occurrence of A, hence the probability of A occurring is $a + b$. In the same way the probability of B occurring is $a + c$. The data of the problem, then, are

$$\begin{aligned} a + b + c + d &= 1, \\ a + b &= p_1, \\ a + c &= p_2, \\ d &= p_3; \end{aligned}$$

and we are to find a. The result is $a = p_1 + p_2 + p_3 - 1$.

[*L. M. Hoskins.*]

249

CUT the two edges AB, CD of the tetraedron $ABCD$ by the plane Π in P, Q respectively; and take P', Q' the harmonic conjugates to P, Q relative to AB, CD. Draw a plane Σ through $P'Q'$ and let M, N be the points in which it cuts AC, BD. Then will the join MN intersect both PQ and $P'Q'$; and in points R, R' which divide MN harmonically. [*E. H. Moore, Jr.*]

SOLUTION.

In fact, the three lines AC, BD, PQ determine an hyperboloid of one sheet, of which one set of generators is the series of lines lying across these three lines, which belong to the second set of generators. Every line of one set intersects every line of the other set, and the lines of one set intersect the lines of the other set in projective ranges of points. ABP, CDQ are evidently lines of the first set, and the line of the second set crossing ABP at P', the fourth harmonic point, would cross CDQ at the fourth harmonic point Q'; that is, $P'Q'$ is a generator of the second set, and every line crossing AC, BD, PQ crosses $P'Q'$ also, and the four points of intersection make an harmonic range; or, what comes to the same thing, every line MN (lying in a plane Σ through and, therefore,) intersecting $P'Q'$ at R', and also intersecting AC, BD at M, N, intersects also PQ at, say, R, and the four points $MNRR'$ make an harmonic range.

To obtain Ex. 207, let Π be the plain at infinity, when the fourth harmonic point to any two points and the point of intersection of the join-line with the plane Π will be simply the middle point of the segment joining the two points.

[*E. H. Moore, Jr.*]

250

A HOMOGENEOUS sphere rests on another such sphere of equal mass, which rests on a table. Everything being smooth and the system being slightly shaken, show that the spheres will separate when the upper one has turned through the angle $\cos^{-1}(\sqrt{3} - 1)$. [*Frank Morley.*]

SOLUTION.

Take the origin at the centre of the lower sphere at the beginning of motion, and a horizontal line through this centre in the plane of motion for axis of x. Let x, y; x', 0 be the co-ordinates of the centres of the spheres, φ the angle the line joining the centres makes with the vertical, R the reaction between the spheres, m the mass of a sphere, and a its radius. The equations of motion in x are

$$m\frac{d^2x}{dt^2} = R \sin \varphi, \qquad m\frac{d^2x'}{dt^2} = - R \sin \varphi.$$

From these equations we have $x = - x'$. The positions of the spheres furnish the relations

$$x = a \sin \varphi, \qquad y = 2a \cos \varphi,$$

$$x' = - a \sin \varphi.$$

The values of the derivatives, $\frac{dx}{dt}, \frac{dx'}{dt}, \frac{dy}{dt}$, and that of the *force function*, $2agm(1 - \cos \varphi)$, give, by the principle of *vis viva*,

$$(2 - \cos^2 \varphi)\, a^2 \frac{d\varphi^2}{dt^2} = 2ag(1 - \cos \varphi).$$

Resolving the forces along the line joining the centres of the spheres, since $2a$ is constant, we have,

$$2am \frac{d\varphi^2}{dt^2} = gm \cos \varphi - R.$$

At the instant the spheres separate $R = 0$; and substituting the value of $a^2 \frac{d\varphi^2}{dt^2}$ in the equation of *vis viva*, the result is

$$\cos^3 \varphi - 6 \cos \varphi + 4 = 0.$$

The roots of this equation are

$$+ \sqrt{3} - 1, \qquad - \sqrt{3} - 1, \qquad + 2.$$

Only the first root can be used, and hence the result. If the lower sphere were fixed the spheres would separate when $\cos \varphi = \frac{2}{3}$. [*A. Hall.*]